APPROXIMATE STATISTICAL THEORY OF A FLUIDIZED BED

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Under the conditions of developed fluidization there are intensive fluctuations both in the fluidizing medium and in the dispersed solid phase. These motions have a decisive effect on the rheological properties of the fluidized bed, and on the chemical reactions and transport processes taking place in it [1]. Thus, for example, in the experiments of Wicke and Fetting [2], who investigated the heat transfer between a fluidized bed and the walls of a heated container, the effective heat transfer coefficient was found to be higher by an order of magnitude than the corresponding result for a fluidized bed held down by a wire grid so that the random motion of the solid phase was reduced. It is clear that the initial stage of any study of the structure of the fluidized bed as a whole, and of the subsequent development of any model, must involve an investigation of local structural properties, including the above fluctuations.

The time variation of the individual particle velocities is due to two different causes. First, there is the interaction between the particles both through direct collisions and through the medium of the liquid phase, and, secondly, there is the interaction with the viscous fluid. These two factors are not independent, so that the set of fluidized particles has certain features characteristic for both a dense gas, with a potential intramolecular interaction, and a set of particles executing Brownian motion in a continuous medium.

Any detailed statistical theory of a system of fluidized particles must be based on a representation of the random particle motions in the medium by a stochastic process with some definite properties (see, for example, [3–4]). Ideally, this theory should lead to the formulation of a transport equation which, in view of the above properties of the system, should have some of the features of both the usual Boltzman transport equation and the Fokker-Planck equation. The solution of this final equation is, of course, more difficult than the solution of the Boltzman or Fokker-Planck equations. Moreover, there is also the problem of applying this equation to different special cases. An alternative approach is to develop an approximate, but still sufficiently effective, theory of the local properties of the fluidized bed, which would combine relative simplicity in application with sufficient rigor and generality. This kind of theory is put forward in the present paper. The conclusions to which it leads are in good qualitative agreement with experiment.

§1. MODEL OF A FLUIDIZED BED

Consider the volume of a fluidized bed containing a very large number of particles, so that its macroscopic characteristics can be regarded as independent of the coordinates. Moreover, let us confine our attention to times t for which these parameters remain practically constant. If the linear dimensions of the chosen volume are much greater than the characteristic spatial scale of the microstructure of the bed, and the time t is much longer than the characteristic time scale for internal motions, then, by analogy with kinetic theory, the fluidized bed is said to be in local equilibrium. If the macroscopic parameters are entirely independent of the coordinates and of time, the system is said to be in a state of equilibrium. Only such states will be discussed below.

If we neglect fluctuational motion of the two phases, the state of the fluidized bed can be described by specifying a single variable, for example, the particle concentration ρ_0 in the layer. The other macroscopic variables such as, for example, the mean velocity u_0 of the liquid phase or the filtration velocity $u^\circ = \varepsilon_0 u_0 = (1 - \rho_0)u_0$ are functions of ρ_0 and of the material constants. Let $f = f(\rho_0, u_0)$ be the interaction force between the phases per unit volume of the fluidized bed.

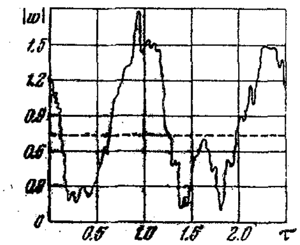

Fig. 1

In the set of coordinates in which the mean velocity of the solid phase is zero, the velocity u_0 is antiparallel to the force of gravity and satisfies the equation

$$f(\rho_0, u_0) = \rho_0 (d_2 - d_1) g, \tag{1.1}$$

where g is the gravitational acceleration, and d_1 and d_2 are the densities of the liquid and of the particle material, respectively. We note that the force f represents the true resistance of the fluidized bed to the flow of the fluidizing medium and, therefore, should be determined, for example, from the observed pressure drop in the fluidized bed.

In reality, the particles in a fluidized bed are in a state of random motion characterized by a continuous velocity distribution, and the true local values of ρ and u differ from the mean values ρ_0 and u_0 about which they fluctuate [1]. The disordered motion of the individual particles, which is similar to the motion of molecules in a gas, has frequently been observed experimentally (see, for example, the review in [5]). The space scale of such motions is, of course, of the order of the mean free path λ of the particles between collisions, while the time scale τ is of the order of $\lambda (w^\circ)^{-1}$ where w° is the mean value of the modulus of the velocity of this motion. There is considerable experimental evidence that additional vertical fluctuations are superimposed on these small-scale and, in the first approximation, isotropic motions. These additional fluctuations have associated spatial Λ and time T scales which are much greater than λ and τ. Figure 1 shows the modulus of the vertical component of the total fluctuation velocity of the particles as a function of time. This curve was obtained by Toomey and Johnstone [6] who investigated the behavior of glass balls $\approx$ 0.38 mm in diameter in air. It is clear from Fig. 1 that, in addition to small-scale fluctuations, for which the time scale τ is of the order of a small fraction of a second, there is also a large-scale fluctuation with a time scale T $\sim$ 1 sec (the dashed line in Fig. 1 corresponds to the filtration velocity for air). The presence in the fluidized bed of relatively large groups of particles undergoing disordered vertical motion, which are phenomenologically analogous to eddies in a turbulent fluid, has also been noted by Bondareva and Todes [7] who introduced the concept of the effective displacement length which is necessary for these groups to decay. According to their estimates, this length is of the order of a few centimeters. Similar and very detailed data obtained by radioactive tracer techniques were recently reported in [8, 9]. The well-defined alternation of fast and slow motions of groups of particles in the vertical direction was also noted in [10] and in many other papers [1].

Analysis of experimental results suggests the following picture of internal motions in a fluidized bed.

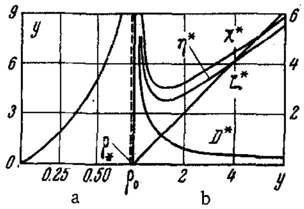

Fig. 2

The individual particles execute random isotropic fine-scale motion which differs from the motion of gas molecules only in that the velocity of each particle may change not only as a result of collisions between the particles, but also as a result of viscous dissipation of their energy. This type of motion leads to fluctuations of the various variable parameters characterizing the state about their mean values, including fluctuations in ρ or the particle-number density $n = \rho/v^\circ$, where v° is the volume of a particle.

It is natural to try to relate the observed large-scale vertical motion of the particles with fluctuations in ρ or n. In fact, the resistance f offered by the particles to the flow of the fluidizing medium will vary within the limits of the volume of the fluidized bed occupied by the fluctuations, and the balance of forces described by Eq. (1.1) will be violated. As a result, the particles participating in the fluctuations will be accelerated either upward or downward under the combined action of the force f and the gravitational force.

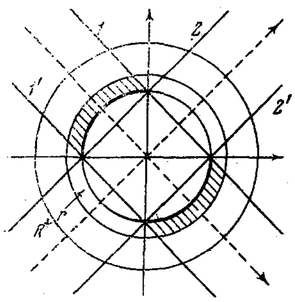

Fig. 3

During the motion and decay of the fluctuational formations, the energy assumed by the particles participating in the large-scale fluctuational motion will be redistributed among the various degrees of freedom of the small-scale random motion, both through collisions and through indirect interactions between the particles in the random pressure field of the liquid phase. This compensates the viscous energy dissipation of the small-scale fluctuational motions. In other words, there is an energy flux E from the mean motion of the liquid phase to the large-scale vertical fluctuations and then to the small-scale motion. In the quasi-stationary state this flux is equal to the power W dissipated into heat as a result of viscous friction. The quasi-stationarity condition

$$E = W \tag{1.2}$$

occupies a central position in the theory given below and shows that the statistical characteristics of a system of particles can vary only as a result of a change in the macroscopic parameters which take place in a time which is much greater than the characteristic times τ and T. It is clear that the equilibrium, or the local equilibrium, defined above is necessarily quasi-stationary. We note that under the conditions of developed turbulent fluidization, or if the fluidizing medium is poorly distributed in the bed, there is an additional energy flux E' due to the capture of the particles by eddies or fluctuations in the liquid phase. Bubbles rising in the bed in the case of inhomogeneous fluidization may play an analogous role. To take these phenomena into account we must introduce a random turbulent field which is unknown a priori, we must consider the motion of the bubbles, etc. All this complicates the problem considerably. We shall therefore ignore these phenomena in the present paper, and this can obviously lead to errors when the theory is applied to highly turbulent fluidization.

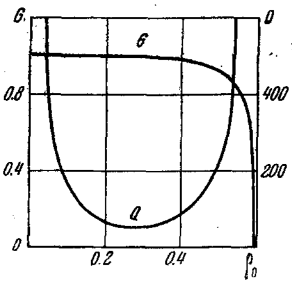

Fig. 4

When the quasi-stationarity condition given by Eq. (1.2) is satisfied, the true ensemble of fluidized particles undergoing velocity changes between collisions and continuously dissipating their energy is statistically equivalent to a fictitious ensemble of particles whose velocities change only as a result of collisions. This hypothesis corresponds to a fact which is well known in kinetic theory, namely, that the parameters which characterize the statistics of equilibrium states are independent of the method whereby the state is established and of the specific form of interaction between the particles.

We shall assume in addition that

$$\tau \ll \tau^\circ$$

where τ° is the relaxation time for a particle moving in a viscous fluid. When Eq. (1.3) is satisfied, the mean free path is, in fact, independent of the interaction between the particles and the medium. It is clear that the above condition is satisfied in many cases which are of interest in practice. Exceptions to this occur where either the liquid phase is highly viscous or the particles are very small. When Eq. (1.3) is satisfied we can analyze transport processes in the fluidized bed system as in the elementary kinetic theory of gases, i.e., we can use the idea of an equivalent ensemble. Strictly speaking, this equivalence principle need not be introduced, and the statistics of the fluidized particles can be developed independently by analogy with the Gibbs distributions using Eqs. (1.2) and (1.3).

If $P(w_i)$ is the particle-velocity distribution function for the small-scale motion normalized to the particle-number density, and $Q(\delta w)$ is the probability that a particle will enter a fluctuation which corresponds to a velocity δw of large-scale motion, then the mean number of particles with velocities in the range $\{dw_i\}\ d(\delta w)$ is given by

$$dn = P(w_i)\,Q(\delta w)\,\{dw_i\}d(\delta w_i),\ \delta w_i = \delta w \delta_{i1}, \quad (1.4)$$

where the $x \sim x_1$ axis lies in the direction of the filtration velocity.

Thus, local fluctuational motions of the particles in the fluidized bed have a very complicated character. They form a superposition of uncorrelated small-scale isotropic motions similar to the thermal motion, and large-scale vertical fluctuations which introduce an anisotropy into the total distribution given by Eq. (1.4).

§2. FORMAL STATISTICS OF EQUILIBRIUM STATES

It follows from the above account that the basic hypotheses are the assumed isotropy of the equilibrium distribution $P(w_i)$ and the assumed independence of the number of particles with a given velocity w on the interaction between the particles. Together with the quasistationarity condition (1.2), these hypotheses lead to the following functional equation:

$$P(w_1^2)\,P(w_2^2) = P(w_3^2)\,P(w_4^2),$$

$$w_1^2 + w_2^2 = w_3^2 + w_4^2,$$

where the subscripts represent the velocities of two different particles before and after interaction.

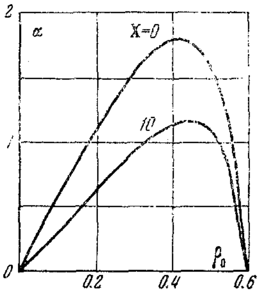

Fig. 5

The solution of this problem leads to the Maxwellian distribution

$$dP(w_i) = n\left(\frac{m}{2\pi\theta}\right)^{3/2}\exp\left(-\frac{mw^2}{2\theta}\right)\{dw_i\}\ , \quad (2.1)$$

where m is the particle mass and θ is a scalar quantity which depends on the macroscopic parameters of the fluidized layer and can be interpreted as the effective statistical "temperature" of the system of fluidized particles.

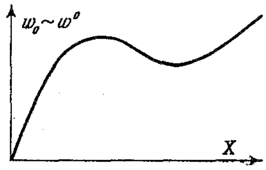

Fig. 6

We note that, in addition to translational degrees of freedom, each particle will also have rotational degrees of freedom. The rotation of individual particles has been observed experimentally [10]. When the radius of the particles is sufficiently small, one would expect that the dissipation of rotational energy occurs much more rapidly than the excitation of these degrees of freedom, so that the mean energy of translational motion is much greater than that of the rotational motion (in other words, the effective rotation temperature is much less than θ). Although the rotation of the particles can, in principle, be taken into account by introducing the Eucken corrections [11], this effect will not be taken into account here.

Using the equivalent ensemble hypothesis, and applying a standard procedure, it is quite easy to obtain for the equilibrium state of the fluidized bed the same results as for a gas. Thus, we can readily evaluate the partition function

$$Z = \frac{1}{N!} V_0^N (2\pi m\theta)^{3/2N},$$

where V_0 is the free volume (in the sense of van der Waals) corresponding to N particles. The effective Helmholtz free energy is then given by

$$\Psi = -\theta \ln Z = -$$

$$-\theta N\ [\ln(V_0 / N) + {}^3/_2\,(\ln\theta + \ln(2\pi\, m))]\,.$$

The effective "entropy" S and the isotropic "pressure" of the solid phase are, respectively, given by

$$S = -\partial\Psi/\partial\theta,\ \ p = -\partial\Psi/\partial V.$$

The relation between the total volume V occupied by the N particles and V_0 can most simply be established by using the equation of state of the ensemble found independently. For a dense gas of spherical particles of radius a we have [11]

$$pv\,(1-\sigma) = \theta,$$

$$\sigma = (v_*/v)^{1/3} = (n/n_*)^{1/3} = (\rho/\rho_*)^{1/3}, \quad (2.2)$$

where $v = V/N$ is the specific volume of a particle, and v_*, n_*, and ρ_* are the values of the corresponding parameters in the closely packed state. Equation (2.2) is valid for ρ not too small, in which case an individual particle will not readily escape from the cell occupied by it. According to [11] this equation is valid for $0.125\rho_* \leq \rho \leq \rho_*$.

If we determine p from Eq. (2.2) and express it in terms of the free energy, we obtain

$$\frac{V}{V_0}\frac{dV_0}{dV} = \frac{1}{1-\sigma}\,,\ \text{ on }\quad V_0 = (V^{1/3} - V_*^{1/3})^3\,.$$

Using Ψ once again, we obtain the following expression for the entropy:

$$S = N\ [3\ln\ (v^{1/3} - v_*^{2/3}) +$$

$$+ {}^3/_2(\ln\theta + \ln\ (2\pi m) + 1)] \quad (2.3)$$

From the results of Enskog [33] on the transport of elastic spheres with delta-function interaction in dense gases we have the following expressions for the shear

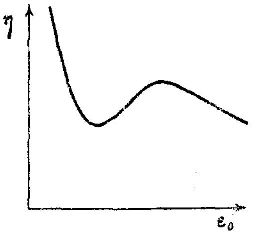

Fig. 7

viscosity η, the bulk viscosity ζ, the self-diffusion coefficient (D) of the particles, and the transport of their small-scale pulsational energy (γ) which is the analog of the thermal conductivity:

$$\eta = 4\rho\eta^\circ (Y^{-1} + 0.8 + 0.76\ Y), \quad \zeta = 4\rho\eta^\circ Y,$$

$$D = 4\rho D^\circ Y^{-1}, \quad \gamma = 4\rho\gamma^\circ \quad (Y^{-1} + 1.2 + 0.75\ Y),$$

$$\eta^\circ = \frac{5}{64a^2}\left(\frac{m\theta}{\pi}\right)^{1/2}, \qquad D^\circ = \frac{3}{32a^2}\frac{v^\circ}{\rho}\left(\frac{\theta}{\pi m}\right)^{1/2},$$

$$\gamma^\circ = \frac{5}{2}\frac{c_v\eta^\circ}{m} = \frac{15}{4}\frac{\eta^\circ}{m},$$

$$Y(\rho) = \frac{pv}{\theta} - 1 \approx \frac{\sigma}{1-\sigma}, \tag{2.4}$$

where η°, D°, γ° are the values of the corresponding coefficients at zero pressure p (in a diluted gas of the same particles).

In the expression given by Eq. (2.4), c_V is the effective "heat capacity" of a single particle at the temperature θ.

The quantity Y is plotted as a function of ρ in Fig. 2a (for $\rho_* = 0.74$, which corresponds to hexagonal close packing). The reduced coefficients η^*, etc. $\eta^* = (4\rho\eta^\circ)^{-1}$ are shown in Fig. 2b [11].

For values of ρ for which Eq. (2.2) is satisfied, the function Y can be calculated from the same equation.

We note that the quantities η°, D°, γ° are independent of the density of the gas. This is a well-known result in the kinetic theory of gases, and is explained by the fact that the reduction in the number of particles passing through an element of area per unit time is exactly compensated by the increase in the mean free path λ as the pressure is reduced. There is an analogous situation in a fluidized bed if the inequality given by Eq. (1.3) is satisfied.

For the probability of a small fluctuation $\delta\rho$ accompanied by a change δS in the entropy we have

$$dQ\,(\delta\rho) \sim \exp\,(\delta S\,(\delta\rho))\,d\,(\delta\rho). \tag{2.5}$$

In particular, if we use Gibbs' first lemma for the relative square of the fluctuation in the number of particles in a volume V of the fluidized bed, we obtain in the usual way

$$\left\langle\left(\frac{\delta N}{N}\right)^2\right\rangle = \frac{\theta}{V}\left|\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_\theta\right| \approx \frac{1}{N}\frac{(1-\sigma)^2}{1-{}^2/_3\sigma}. \tag{2.6}$$

In deriving this result we have used the equation of state in the form given by Eq. (2.2). The expression given by Eq. (2.6) was derived in [12] on the basis of the well-known Smoluchowski formula, but this is incorrect because the latter formula is not valid for very dense systems such as a fluidized bed.

§3. SMALL FLUCTUATIONS IN A SET OF FLUIDIZED PARTICLES

When the volume V is sufficiently large we can assume that N ≈ const, in accordance with Eq. (2.6). We shall suppose that, as a result of fluctuations, the mean distribution of particles in this volume, $\rho = \rho_0 =$ const, has become $\rho = \rho_0 + \varphi\,(x, y, z, t)$. The form of the function φ is restricted by the conditions

$$-\rho_0 \leqslant \varphi \leqslant \rho_* - \rho_0, \qquad \int_V \varphi\,dV = 0. \tag{3.1}$$

Let us now introduce the quantities ψ and $\psi^\pm$ by the formulas

$$V\psi = \int_V \varphi^2\,dV, \qquad V\psi^\pm = \int_{V\pm} \varphi^2\,dV^\pm,$$

where V^+ and V^- are the regions of V in which φ is positive and negative, respectively. Analysis of arbitrary fluctuations in this system is very difficult (for example, $\delta S(\varphi)$ in Eq. (2.5) is, in general, a complicated functional of φ). To obtain approximate results we shall use the assumption that the fluctuations are small. The state corresponding to different distributions ρ (x, y, z, t) with given intensity ψ corresponds to a change of entropy (as compared with the undisturbed state $\rho = \rho_0$) which can be written in the form

$$\delta S = 3\int_V \delta\,[n\ln(v^{1/3} - v_*^{1/3})]\,dV =$$

$$= \frac{3}{v^0}\int_V \delta\left[\rho\ln\left(v^{\circ\,1/3}\frac{\rho_*^{1/3} - \rho_0^{1/3}}{\rho_0^{1/3}\rho_*^{1/3}}\right)\right]dV,$$

where we have used Eq. (2.3).

From Eq. (3.1) we have for small fluctuations

$$\delta S \approx -\frac{1 - {}^2/_3\sigma}{2\rho_0^2(1-\sigma)^2}N\psi = -Q\,(\rho_0, \rho_*)\,N\psi. \tag{3.2}$$

To obtain the probability of the disturbed state corresponding to a given value of ψ in accordance with Eq. (2.5), we must find its statistical weight, i.e., introduce the measure of the set of functions φ(x, y, z, t) for which the quantity ψ lies in a given interval. Let us represent φ by the trigonometric polynomial

$$\varphi = \sum_{i,j,l} a_{ijl}(t)\cos\frac{2\pi ix}{L_x}\cos\frac{2\pi jy}{L_y}\cos\frac{2\pi lz}{L_z} =$$

$$= \sum_{\mathbf{m}}^{M} a_{\mathbf{m}}(t)\prod_k \cos\frac{2\pi m_k x_k}{L_k}$$

$$M = M_x M_y M_z, \quad x_k = x,\ y,\ z\ . \tag{3.3}$$

We then have the Parseval equation

$$V\psi = \frac{L_x L_y L_z}{8}\sum_{\mathbf{m}}^{M} a_{\mathbf{m}}^2(t) = V\frac{r^2}{8}. \tag{3.4}$$

In these expressions L_x, L_y, L_z are the linear dimensions of the volume V, $1 \le i \le M_x$, $1 \le i \le M_y$, $1 \le l \le M_z$, and the vector **m** represents different combinations of the subscripts i, j, and l. It is readily seen that functions of the form given by Eq. (3.3) automatically satisfy the second condition in Eq. (3.1). Transforming from V to V^+, we shall write $\varphi^+ \ge 0$, on V^+ in the form

$$\varphi^+ = \sum_{\mathbf{m}}^{M} a_{\mathbf{m}}^+(t) \prod_k \cos\frac{2\pi m_k x_k^+}{L_k^+}, \qquad \psi^+ = \frac{(r^+)^2}{8}.$$

Since φ^+ must be positive in V^+, we have $a_{\mathbf{m}}^+ \geq 0$. It is also readily shown that, since φ^+ must be bounded, we have

$$\sum a_{\mathbf{m}}^+(t) \leqslant \rho_* - \rho_0.$$

The quantities $a_{\mathbf{m}}^+$ (t) can be regarded as the coordinates of a point in M-dimensional space. The possible values of $a_{\mathbf{m}}^+$ define the region of this space bounded by the hypersurfaces

$$\sum (a_{\mathbf{m}}^+)^2 = (R^+)^2, \qquad \sum a_{\mathbf{m}}^+ = \rho_* - \rho_0, \qquad a_{\mathbf{m}}^+ = 0, \quad (3.5)$$

where R^+ is the maximum value of r^+ (referred to the entire volume V) which is defined below. The required measure can reasonably be taken to be the volume of the spherical layer (r, r + dr) bounded by the hypersurfaces (3.5). This choice is dictated by the statistical homogeneity of the above space, i.e., the equivalence of different coefficients $a_{\mathbf{m}}^+$, but it also follows from Eq. (3.4). It is readily seen that instead of the volume of the spherical layer in the single quadrant $a_{\mathbf{m}}^+ \geq 0$ we can take the volume of the layer bounded by the 2^M hyperplanes:

$$\sum a_{\mathbf{m}}^+ (-1)^{s_{\mathbf{m}}} = \rho_* - \rho_0. \qquad (3.6)$$

In this expression the superscripts $s_{\mathbf{m}}$ can assume values between 0 and 1 independently of each other.

Consider to begin with a volume $d\Omega_M$ bounded by the M-dimensional spheres r and r + dr and two hyperplanes of the form (3.6) which are symmetric relative to the origin. We have

$$d\Omega_M = S_M'(r)\,dr = \omega_M S_M(r)\,dr,$$

where S_M' is the area of the corresponding spherical zone, S_M is the area of the entire hypersphere of radius r, and ω_M is, by definition, equal to S_M'/S_M. This is illustrated by Fig. 3 which shows the above region for M = 2. Lines 1 and 1' correspond to the symmetric hyperplanes, and the circle R^+ corresponds to the hypersphere. The equivalent of S_M' is the total length of the thick arc shown in the figure, while the hatched areas are equivalent to the volume of the spherical layer.

Let us now define in the space of the Fourier coefficients a unitary transformation $\{a_{\mathbf{m}}^+\} \to \{b_{\mathbf{m}}^+\}$ such that the coordinates of the points of intersection of the perpendicular dropped onto the hyperplane from the origin are $b_{\mathbf{m}0}^+ = \pm b\delta_{\mathbf{m}1}$ (broken line in Fig. 3). To determine b we shall use the symmetry of $a_{\mathbf{m}}^+$ in Eq. (3.6). The coordinates of the above point in the basis $\{a_{\mathbf{m}}^+\}$ are then given by

$$a_{\mathbf{m}0}^+ = \pm M^{-1}(\rho_* - \rho_0).$$

Hence, we find that

$$b = \pm\left(\sum (a_{\mathbf{m}0}^+)^2\right)^{1/2} = \frac{1}{\sqrt{M}}(\rho_* - \rho_0).$$

It is clear that, as M increases, the hyperplanes approach each other without limit. The ratio of the area of the spherical zone to the area of the entire sphere can be shown to be [13]

$$\omega_M(r) \approx \operatorname{erf}\left(\frac{\rho_* - \rho_0}{\sqrt{2r}}\right), \qquad M \gg 1.$$

Let us now introduce two new symmetric hyperplanes (2 and 2' in Fig. 3), and consider their intersection with the figure bounded by the hypersphere r and the original hyperplanes. Since for M ≫ 1, these hyperplanes can be as close to each other as desired, and we can assume that the area of the M-dimensional annular zone is proportional to the area of the (M − 1)-dimensional sphere of radius r, so that we can repeat the above analysis for the (M − 1)-dimensional sphere. We thus find by induction that

$$\omega_{M-n} \approx \omega_{M-n+1} \approx \ldots \approx \omega_M, \qquad M - n \gg 1.$$

Hence, apart from unimportant constant factors (we recall that $S_M \sim r^{M-1}$), the required measure in V^+ is given by

$$\chi^+(r)\left[r \operatorname{erf}\left(\frac{\rho_* - \rho_0}{\sqrt{2r}}\right)\right]^M, \qquad (3.7)$$

where $\chi^+(r)$ is a function of r which is independent of M.

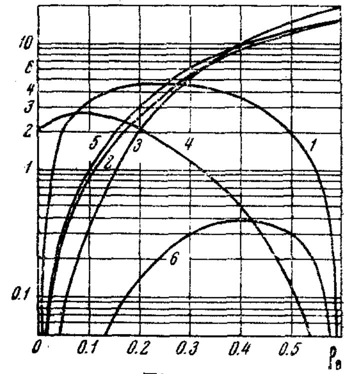

Fig. 8

A similar analysis is valid for φ^- in the region V^-. The corresponding measure is then again given by Eq. (3.7) except that $\rho_* - \rho_0$ is replaced by ρ_0.

From the above results we find that the probabilities $dQ^\pm(r)$ for the quantities $r^\pm$ to lie in the interval (r, r + dr) are given by

$$dQ^+(r) \sim \chi^+(r)\left[r \operatorname{erf}\left(\frac{\rho_* - \rho_0}{\sqrt{2r}}\right)\right]^M \exp\left(-\frac{Q}{8} Nr^2\right) dr,$$

$$dQ^-(r) \sim \chi^-(r)\left[r \operatorname{erf}\left(\frac{\rho_0}{\sqrt{2r}}\right)\right]^M \exp\left(-\frac{Q}{8} Nr^2\right) dr.$$

We must now find M and $R^\pm$. The Fourier coefficients $a_{\mathbf{m}}(t)$ correspond to different degrees of freedom of the set of functions φ in Eq. (3.3). Suppose that the particles move in groups of N_0 particles per group. This means that by specifying the function φ we are also specifying N/N_0 vector functions of time defining the position of these groups in the volume V. Instead of these functions we can introduce the set of independent scalar coefficients $a_{\mathbf{m}}(t)$ and, since they are independent, we have $M = 3N/N_0$.

In the limit of zero pressure we have $N_0 = 1$, i.e. each particle behaves independently of other particles. In a dense system this is no longer true, since a virtual displacement of a given particle gives rise to a regrouping and a correlated displacement of neighboring particles. The limiting case when the specification of the motion of a single particle completely defines the motion of all other particles in a given group occurs in the case of close packing of the particles in this group. We shall suppose that close packing is achieved in a volume $V' \ll V$ of the fluidized bed in which the mean number of particles is $N' = V'/v = (\rho_* / v^\circ)V'$. In the closely-packed state this volume will contain $N_0 = (\rho_* / \rho_0)N'$ particles, and hence it follows that

$$\delta N' = \rho_0^{-1}(\rho_* - \rho_0)N_0.$$

We can then show that for closely packed groups of particles

$$dq(N_0) \sim \exp(-\alpha N_0)\,dN_0, \qquad \alpha = \frac{(\rho_* - \rho_0)^2}{2}\frac{\rho_*}{\rho_0^3}\frac{1 - {}^2/{}_3\varsigma}{(1-\varsigma)}.$$

From this distribution we have the following expressions for the averages $\langle N_0\rangle$ and $\langle M\rangle$:

$$\langle N_0\rangle = \frac{1+\alpha}{\alpha} = \frac{1}{G},$$

$$\langle M\rangle = 3N\alpha e^\alpha(-\operatorname{Ei}(-\alpha)) \approx \frac{3N}{\langle N_0\rangle} = 3GN. \qquad (3.9)$$

This expression for $\langle M\rangle$ should be used in all calculations involving the representation of the function φ by Eq. (3.3). Figure 4 shows a plot of Q as given by Eq. (3.2) and G as given by Eq. (3.9) as functions of ρ_0 for $\rho_* = 0.6$. It is clear from this figure that G is appreciably different from unity only in a small neighborhood of ρ_*.

The maximum values of R^+ and R^- are reached when the region V^- of the volume V is completely free of particles, and V^+ is filled with closely packed particles. Using Eqs. (3.1) and (3.4) we can show that

$$R^+ = \max\{r^+\} = \left(8\frac{\rho_0}{\rho_*}(\rho_* - \rho_0)^2\right)^{1/2},$$

$$R^- = \max\{r^-\} = \left(8\frac{\rho_* - \rho_0}{\rho_*}\rho_0^2\right)^{1/2}. \quad (3.10)$$

Consider the integral

$$J_k^+ \sim \int_0^{R^+} \chi^+(r)\, r^k \left[r \operatorname{erf}\left(\frac{\rho_* - \rho_0}{\sqrt{2r}}\right)\right]^{\langle M\rangle} \times$$
$$\times \exp\left(-\frac{Q}{8}Nr^2\right) dr\,,$$

where R^+ is given by Eq. (3.10).

The integral reaches a maximum at $r = r_m^+$ where

$$(r_m^+)^2 = 12\frac{G}{Q}\left[1 - \frac{\rho_* - \rho_0}{\sqrt{\pi}}\times\right.$$
$$\left.\times \exp\left(-\frac{(\rho_* - \rho_0)^2}{2(r_m^+)^2}\right)\operatorname{erf}^{-1}\left(\frac{\rho_* - \rho_0}{\sqrt{2}r_m^+}\right)\right]. \quad (3.11)$$

There is an analogous equation for r_m^- except that $\rho_* - \rho_0$ is replaced by ρ_0. The asymptotic representation of the integrals $J_k^{\pm}$ for large N can be obtained by the Laplace method:

$$\langle (r^{\pm})^2\rangle = (\min\{r_m^{\pm}, R^{\pm}\})^2, \quad \langle r^2\rangle = \langle (r_m^+)^2\rangle + \langle (r_m^-)^2\rangle\,.$$

In practically the entire region of variation of ρ_0 we have $r_m^{\pm} < R^{\pm}$. This supports the validity of the theory of small fluctuations used above.

Neglecting the difference between G and unity, and between $r_m^{\pm}$ and $12G/Q$, we finally have

$$\langle\psi\rangle = {}^1/_8\langle r^2\rangle \approx 3Q^{-1}\,. \quad (3.12)$$

We shall use this expression in the calculations below.

§4. EFFECTIVE TEMPERATURE, INTENSITY OF FLUCTUATIONS, AND TRANSPORT COEFFICIENTS

We shall use only the first two terms in the expansions of the different functions in terms of φ in the neighborhood of ρ_0. In fact, many important functions are very dependent on ρ_0 and, therefore, the results which we shall obtain below will be valid only to within an order of magnitude.

To be specific, we shall suppose that the forces acting on the particles undergoing the fluctuational motion can be satisfactorily described by the semi-empirical formula [14],

$$R = \frac{A\varepsilon^{3.75}}{18 + 0.6(A\varepsilon^{4.75})^{1/2}}\,,$$

$$R = \frac{2u_0 a}{\nu_0}\,, \qquad A = \frac{8ga^3}{\nu_0^2}\,\frac{d_2 - d_1}{d_1}\,, \quad (4.1)$$

where R and A are the Reynolds and Archimedes numbers, and ν_0 is the kinematic viscosity of the liquid phase. In the limiting case of very small Archimedes numbers, this formula is confirmed independently by theoretical considerations [15]. It follows from Eq. (4.1) that the additional force $F(\varphi)$ acting on a particle in its large-scale motion is given by

$$F(\varphi) \approx 6\pi\mu_0 a\,[K_0(\delta u - \delta w) + K_0' u_0 \varphi],$$

$$K_0 = K(\rho_0, X), \qquad K(\rho, X) = \varepsilon^{-3.75}(1 + \varepsilon^{2.375}X),$$

$$K_0' = \frac{d}{d\rho}K(\rho, X)\big|_{\rho=\rho_0}, \quad (4.2)$$

where we have used the new dimensionless parameter

$$X \approx 0.033A^{1/2}. \quad (4.3)$$

The quantities δu and δw represent the changes in the velocities of the liquid and solid phases within the fluctuations and, in general, are time-dependent. The change δu can be approximately estimated by considering the filtration of a liquid in a porous medium with porosity $1 - \rho_0$ and containing a moving inhomogeneity with porosity $1 - \rho_0 - \varphi$. This can be done by using, for example, the equations of [16]. We shall assume, for simplicity, that the characteristic time for an appreciable change in the flow of the liquid near the inhomogeneity is considerably greater than the mean lifetime T of the inhomogeneity. Hence, assuming that the flow rate of the liquid phase is constant, we have

$$\delta u \approx u_0(1 - \rho_0)^{-1}\varphi\,. \quad (4.4)$$

In the opposite limiting case, it is sufficient to consider stationary filtration. If we use the subscripts 0 and 1 to refer to the exterior and interior of the inhomogeneity, we can write down equations for the pressure in the liquid and for the rate of filtration in terms of pressure gradients in the form

$$\Delta p_i = 0, \qquad \mathbf{u}^{(i)} = -\frac{\varepsilon_i}{\beta_i}\frac{\partial p_i}{\partial \mathbf{r}}\,, \qquad \beta(\rho) = \frac{9}{2}\frac{\mu_0}{a^2}\rho K(\rho, X)\,.$$

We shall seek the solution of the Laplace equations in the form

$$p_0 = (-\beta_0 + Br^{-3})((\mathbf{u}_0 - \delta\mathbf{w})\,\mathbf{r}), \qquad p_1 = -C((\mathbf{u}_0 - \delta\mathbf{w})\,\mathbf{r})\,.$$

The constants B and C can be deduced from the continuity of pressure and of the normal component of the filtration velocity on the boundary of the inhomogeneity. We thus obtain

$$\mathbf{u}_1 - \delta\mathbf{w} = \frac{3\varepsilon_0/\beta_0}{2\varepsilon_0/\beta_0 + \varepsilon_1/\beta_1}\,\frac{\beta_0}{\beta_1}(\mathbf{u}_0 - \delta\mathbf{w})\,.$$

Hence, we can readily find $\delta u = u_1 - u_0$. We shall use δu in the form given by Eq. (4.4), since the results obtained in this way will be valid to within an order of magnitude in other cases also.

Direct solution of Eq. (4.2) for $\delta w(t)$ is difficult. However, if we consider the mean velocity w_0 of the vertical large-scale motion in a volume containing many fluctuational formations, we have from Eqs. (4.2), (4.4), and (3.12)

$$w_0 \approx w_\infty(1 - e^{-T/\tau^\circ}), \qquad w_\infty \approx W_*(\rho_0, X)\,u_0\,,$$

$$W_*(\rho_0, X) = \left(\frac{3}{Q}\right)^{1/2}\left(\frac{\partial \ln K}{\partial \rho}\Big|_{\rho=\rho_0} + \frac{1}{1-\rho_0}\right),$$

$$\tau^\circ = \frac{2\varkappa a^2}{9\nu_0 K_0}\,, \qquad \varkappa = \frac{d_2}{d_1}\,. \quad (4.5)$$

We note that, in the case which we are considering, a positive fluctuation $\varphi > 0$ will also lead to the upward displacement of the particles, in accordance with Eq. (4.5). This confirms the conclusion reported by Leva [5, 17] that the aggregation of particles gives rise to a tendency toward an additional expansion of the fluidized bed. It can be shown that when a stationary filtration state is established near the inhomogeneity, there is a value $\rho_0 = \rho'$ such that for $\rho_0 < \rho'$ the fluctuation $\varphi > 0$ leads to the fall of the aggregated particles in the bed. The value $\rho_0 = \rho'$ corresponds to a sharp minimum of w_0 and of other statistical parameters of the fluidized bed. As the case defined by Eq. (4.4) is approached, the quantity ρ' decreases and eventually vanishes altogether.

Let us now consider the energy relations in the fluidized bed. The work ΔA performed by the forces (4.2) per unit volume per unit time can be written in the form $\Delta A \approx 6\pi\mu_0 a n K_0(w_\infty - w_0)\,w_0$. This work will compensate any loss of energy through large-scale fluctuations

$$E \approx nmw_0^2 / 2T.$$

By Eq. (1.2) the energy E is equal to the energy dissipation W of small-scale fluctuational motions. The latter energy can be written in the form [we are averaging over the distribution Eq. (2.1)]

$$W \approx 18\pi\mu_0 anK_0 \, [1 + {}^1/_2 sa\,(\pi\nu_0\tau)^{-1/2}] \, (\theta / m).$$

The last term in the square brackets appears because the viscous force acting on a particle rapidly brought into motion is different from the force acting on it during uniform motion [18]. The coefficient $s \sim 1$ characterizes the relative magnitude of the sudden change in velocity during collisions and can be evaluated exactly. We shall assume that $s \approx 1/2$. For the parameter τ we have the obvious estimate

$$\tau \approx \lambda\left(\frac{\theta}{m}\right)^{-1/2} \approx a\,\frac{\rho_*^{1/3} - \rho_0^{1/3}}{(\rho_0\rho_*)^{1/3}}\left(\frac{\theta}{m}\right)^{-1/2}.$$

Equating ΔA and E, and bearing in mind Eq. (4.5), we obtain an equation from which we find that $T \approx \approx 1.25\tau^\circ$. Therefore

$$w_0 \approx 0.715\, w_\infty = 0.715\, W_*(\rho_0, X)\, u_0. \qquad (4.6)$$

The ratio α of the velocity w_0 and the filtration velocity u° is shown in Fig. 5 as a function of ρ_0 for $X = 0$ and $X = 10$. It is clear that even the large-scale component of the total vertical velocity may approach the filtration velocity, in accordance with the curve of Fig. 1. The quantity ρ_* was assumed to be 0.6.

The fact that the plots of α versus ρ_0 show a maximum agrees with the experiments. It is well established [1, 7, 19] that the initial increase in the fluctuation velocity with increasing u° is replaced by a reduction u° has reached a critical value.

If we equate E to W, we obtain the equation for the effective temperature of the fluidized particles:

$$\frac{\theta}{m}\left[1 + \gamma\left(\frac{\theta}{m}\right)^{1/4}\right] = \frac{\tau^\circ}{6T} w_0^2$$

$$\gamma \approx \frac{1}{4}\left(\frac{a}{\pi\nu_0}\right)^{1/2}\left(\frac{\rho_0^{1/3}\rho_*^{1/3}}{\rho_*^{1/3} - \rho_0^{1/3}}\right)^{1/2}. \qquad (4.7)$$

It is readily seen that the mean square velocity of small-scale motions is always considerably less than w_0, and $w^\circ/w_0 = (\theta/m)^{1/2}/w_0 < 0.4$. This is in agreement with all the experiments cited above. For example, according to [8] the lateral velocity w° is lower by a factor of 3/4 than the longitudinal velocity $w_0 + w^\circ$. From Eqs. (1.1) and (4.1) we have

$$u_0 = \frac{v^\circ (d_2 - d_1) g}{6\pi\mu_0 a K(\rho_0, X)}.$$

Substituting this into Eq. (4.6), and using Eq. (4.3) and $\varkappa = d_2/d_1$, we obtain

$$w_0 \approx 3.8\,(\varkappa - 1)^{1/3}\, W_0\,(\rho_0, X)\, X^{4/3} w_*,$$

$$W_0 = W_* K^{-1}, w_* = (g\nu_0)^{1/3}. \qquad (4.8)$$

The velocity w° can readily be determined from Eqs. (4.7) and (4.8). The characteristic velocity w_* is proportional to the cube root of the kinematic viscosity of the liquid, and is in general independent of the type of fluidized particles. The parameters of the particles affect the intensity of the local motions only through the dimensionless parameters X and $\varkappa$. Under these conditions $w^\circ \sim w_0 \sim \varkappa^{1/3}$, $\varkappa \gg 1$ and, moreover,

$$w^\circ \sim w_0 \sim X^{4/3} \sim A^{2/3},$$

$$X \ll X_0 \approx \varepsilon_0^{-2.375} \qquad (A \ll A_0)$$

$$w^\circ \sim X^{2/15} \sim A^{1/15}, \quad w_0 \sim X^{1/3} \sim A^{1/6}, \qquad (X \gg X_0).$$

These relations are satisfactorily confirmed by experiments. For example, Borodulya and Tamarin [21] have investigated the effective thermal conductivity of a fluidized bed in the horizontal ($a_1 \sim w^\circ$) and vertical ($a_2 \sim w_0$) directions, and have obtained empirical formulas of the form $a_i \sim A^{\delta_i}$, where the exponents δ_i decrease with decreasing A. According to their estimates, $\delta_1 \leqslant 0.27$, $\delta_2 \leqslant 0.43$ in a wide range of Archimedes numbers (up to $5 \cdot 10^4$). For $A > 5 \cdot 10^4$ they found that $0 < \delta_1 < 0.12$, $0 < \delta_2 < 0.144$ which is in agreement with the above asymptotic estimates. In the case which we have considered, when the formula given by Eq. (4.4) is valid, the quantities w_0 and w° increase monotonically with A even for $A \sim A_0$. It is readily shown, however, that, in the second limiting case (stationary filtration) both these quantities have a maximum and a minimum as functions of A (their dependence on X is illustrated by Fig. 6). This feature of the function $w_0(A)$ and $w^\circ(A)$ is also in agreement with the experimental results. Bondareva* has observed experimentally the fluctuation-velocity maximum in the region of small A ($A \sim A_0$), and explained it correctly by the change in the hydrodynamic state of the flow past the particles in a fluidized bed as A increases.

The expressions given by Eqs. (4.7) and (4.8) completely determine the coefficients in Eq. (2.4) and the isotropic pressure of the solid phase. It can be shown [we are assuming, for simplicity, that $\gamma = 0$ in Eq. (4.7)] that

$$\eta \approx 15\varkappa\, H(\rho_0, X)\, X^2 \mu_0,$$

$$H = \rho_0 W_0 (Y^{-1} + 0.8 + 0.76Y),$$

$$\zeta \approx 15\varkappa\, Z(\rho_0, X)\, X^2 \mu_0, \quad Z = \rho_0 W_0 Y,$$

$$D \approx 6\,\Delta(\rho_0, X)\, X^2 \nu_0, \quad \Delta = W_0 Y^{-1},$$

$$\gamma \approx 3.9 \cdot 10^{-2} (\varkappa - 1)\, \Gamma(\rho_0, X) \quad (g/\nu_0),$$

$$\Gamma = \rho_0 W_0 (Y^{-1} + 1.2 + 0.75Y),$$

$$p \approx 1.9\,(\varkappa - 1)^{2/3}\, \Pi(\rho_0, X)\, X^{8/3} (d_2 w_*^2),$$

$$\Pi = \rho_0 (1 + Y) W_0^2. \qquad (4.9)$$

We note that the coefficients in Eq. (4.9) describe only the isotropic components in the resultant transport of the various quantities, which are connected only with the small-scale motion of the particles. The corresponding anisotropic components which appear as a result of large-scale fluctuational motion can, in principle, be taken into account by analogy with the

*A. K. Bondareva, Motion of Particles and Heat Transfer in a Fluidized Bed. Candidate's Dissertation, Leningrad, 1958. Some of the results given in this work, especially those referring to fluctuational motions of the particles in the bed, can be found in [1].

theory of turbulent mixing by introducing, for example, certain analogs of the Reynolds stresses, the effective displacement path l by analogy with the Prandtl parameter, or the parameter introduced in [7] and equal to $\sim w_0 T$. However, this forms an independent problem. It is clear that an anisotropic term will also appear in the expression for the total pressure of the solid phase.

The kinematic viscosity η has not been corrected for effects due to friction between the particles during their relative motion. The viscosity η' due to this mechanism of energy dissipation becomes very high when $\rho_0 \to \rho_*$, but falls rapidly with decreasing ρ_0, becoming very small for ρ_0 quite close to ρ_*. It is clear that in the region where η' is small, the dependence of the total momentum transfer along the vertical, and consequently of the "longitudinal" viscosity, on ε_0 is of the form shown by Fig. 7. Similar curves were found for the local velocity as a function of the rate of filtration in [22].

Figure 8 shows the quantities W_0, H, Z, Δ, Γ, Π, multiplied by 100 as functions of ρ_0 for $\rho_* = 0.6$ and $X = 10$ (curves 1–6, respectively). These functions do not contradict the data given in the literature.

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REFERENCES

1. S. S. Zabrodskii, Hydrodynamics and Heat Transfer in a Fluidized Bed [in Russian], Gosenergoizdat, 1963.

2. E. Wicke and F. Fetting, Wärmeübertragung in Gaswirbelschichten, Chem. Ingr.-Techn., vol. 26, no. 6, 1954.

3. R. L. Peskin, Some Effects of Particle-Particle and Particle-Fluid Interaction in Two Phase Flow Systems, Proc. Heat Transfer and Fluid Mech, Inst., Stanford Univ. Press, Calif., 1960.

4. V. G. Levich and V. P. Myasnikov, "Kinetic theory of the fluidized state," Khim. prom-st, no. 6, 1966.

5. M. Leva, Fluidization [in Russian], Gostekhizdat, 1961.

6. R. Toomey and H. Johnstone, "Heat transfer between beds of fluidized solids and the walls of container," Chem. Engng. Progr. Symp. Ser., vol. 49, no. 5, 1953.

7. A. K. Bondareva and O. M. Todes, "Heat conduction and heat transfer in a fluidized bed," Inzh.-fiz. zh., vol. 3, no. 2, 1960.

8. N. B. Kondukov, A. N. Kornilaev, I. M. Skachko, A. A. Akhromenkov, and A. S. Kruglov, "A study of the parameters of motion of particles in a fluidized bed by the radioactive tracer method. 1. Experimental methods and trajectory of the particles. 2. Particle kinematics," Inzh.-fiz. zh., vol. 7, no. 7, 1964.

9. A. A. Akhromenkov and A. S. Kruglov, "Determination of the velocity of solid particles in a fluidized bed by the method of labeled atoms," Inzh.-fiz. zh., vol. 10, no. 5, 1966.

10. L. Massimilla and I. W. Westwater, "Photographic study of solid-gas fluidization," Amer. Inst. Chem. Engng. J., vol. 6, no. 1, 1960.

11. J. O. Hirschfelder, C. F. Curtiss, and R. B. Bird, Molecular Theory of Gases and Liquids [Russian translation], John Wiley and Sons, Inc., New York, 1961.

12. V. I. Kislykh, "Fluctuations in the number of particles in a fluidized bed," Inzh.-fiz. zh. [Journal of Engineering Physics], vol. 10, no. 5, 1966.

13. M. Kats, Probability and Allied Problems in Physics [Russian translation], Izd. Mir, 1965.

14. V. D. Goroshko, R. B. Rozenbaum, and O. M. Todes, "Approximate relations in the hydraulics of a fluidized bed and restricted fall," Neft i gaz, vol. 1, no. 1, 1958.

15. Yu. A. Buevich, "Interaction of phases in concentrated dispersed systems," PMTF [Journal of Applied Mechanics and Technical Physics], no. 3, 1966.

16. Yu. A. Buevich, "Two-fluid hydrodynamics of a fluidized bed," Izv. AN SSSR, MZhG [Fluid Dynamics], no. 4, 1966.

17. M. Leva and M. Grummer, "A correlation of solids turnover in fluidized systems, its relation to heat transfer," Chem. Engng. Progr., vol. 48, no. 6, 1952.

18. L. D. Landau and E. M. Lifshits, Mechanics of Continuous Media [in Russian], Gostekhizdat, 1954.

19. E. I. Cairns and I. M. Prausnitz, "Macroscopic mixing in fluidization," Amer. Inst. Chem. Engng. J., vol. 6, no. 4, 1960.

20. A. I. Tamarin and V. A. Borodulya, "A study of the thermal diffusivity (particle mixing) in a fluidized bed with immersed grids," Inzh. fiz. zh., vol. 6, no. 11, 1963.

21. V. A. Borodulya and A. I. Tamarin, "A study of the effective thermal diffusivity in a fluidized bed," Inzh.-fiz. zh., vol. 7, no. 12, 1964.

22. W. W. Shuster and F. C. Haas, "Point viscosity measurements in a fluidized bed," J. Chem. and Engng. Data, vol. 5, no. 4, 1960.

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